

Nonsmooth mappings with Lipschitz shadowing

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Abstract. We study conditions under which a piecewise affine mapping has the Lipschitz shadowing property. As an application, we show that there exists a homeomorphism with a nonisolated fixed point having the Lipschitz shadowing property.

1. Introduction. The theory of shadowing of pseudotrajectories (approximate trajectories) is now a well-developed branch of the theory of dynamical systems (see, for example, the monographs [1, 2] and the recent survey [3]).

Recently, a lot of attention has been paid to dynamical systems having special shadowing properties (Lipschitz and Hölder, see [4 – 6]). In particular, it was shown in [4] that a diffeomorphism having the Lipschitz shadowing property is structurally stable (thus, Lipschitz shadowing property is equivalent to structural stability). The proof in [4] essentially uses the smoothness of the considered dynamical system (the Mañé theorem [7] giving several characterizations of structural stability of diffeomorphisms is applied).

At the same time, it is possible to define the Lipschitz shadowing property for homeomorphisms (and endomorphisms) of a metric space (see below).

Of course, if a homeomorphism is topologically conjugate to a structurally stable diffeomorphism and both the conjugacy and its inverse are uniformly Lipschitz continuous, then the homeomorphism has the Lipschitz shadowing property. In this connection, it is natural to ask: Are homeomorphisms having the Lipschitz shadowing property similar (in a sense) to structurally stable diffeomorphisms?

In this short note, we give an example of a homeomorphism of the segment having the Lipschitz shadowing property and a nonisolated fixed point. This example shows that the answer to the above question is negative.

Let us give the corresponding definitions (for the case of an endomorphism; for a homeomorphism, the definition is literally the same).

Let (M, dist) be a metric space and let $f : M \rightarrow M$ be a continuous mapping (we do not distinguish f and the semi-dynamical system generated by f). As usual, a sequence $\pi = \{p_k \in M; k \in \mathbb{Z}\}$ is called a trajectory of f if

$$p_{k+1} = f(p_k), \quad k \in \mathbb{Z}.$$

Fix a $d > 0$. We say that a sequence $\xi = \{x_k \in M; k \in \mathbb{Z}\}$ is a d -pseudotrajectory of f if

$$\text{dist}(x_{k+1}, f(x_k)) \leq d, \quad k \in \mathbb{Z}. \quad (1)$$

The (standard) shadowing property of f means that, given an $\varepsilon > 0$, we can find a $d > 0$ such that for any d -pseudotrajectory $\xi = \{x_k\}$ of f there is a trajectory $\pi = \{p_k\}$ satisfying the inequalities

$$\text{dist}(x_k, p_k) \leq \varepsilon, \quad k \in \mathbb{Z}. \quad (2)$$

Finally, we say that f has the Lipschitz shadowing property if there exist $\mathcal{L}, d_0 > 0$ such that for any d -pseudotrajectory ξ of f with $d \leq d_0$ there is a trajectory π satisfying inequalities (2) with $\varepsilon = \mathcal{L}d$.

The structure of the paper is as follows. In Sec. 2, we prove a general sufficient condition under which a “piecewise affine” mapping of \mathbb{R}^n has a “conditional” Lipschitz shadowing property (this means that only pseudotrajectories satisfying some additional assumptions are shadowable). In Sec. 3, we construct the above-mentioned example of a homeomorphism of the segment having the Lipschitz shadowing property and a nonisolated fixed point (and apply to it the result of Sec. 2).

2. Conditional shadowing result. To simplify presentation, we consider a Lipschitz continuous mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with Lipschitz constant L_0 (without loss of generality, we assume that $L_0 \geq 1$) for which there exists a family of sets $G_l \subset \mathbb{R}^n, l \in \Lambda$, with disjoint interiors such that the following conditions hold.

First, for any $l \in \Lambda$ we fix complementary orthogonal linear subspaces S_l and U_l of \mathbb{R}^n (let their dimensions be s_l and u_l , respectively) with coordinates $\xi \in S_l$ and $\eta \in U_l$ and denote

$$N(\Delta, p) := \{p + (\xi, \eta) : |\xi|, |\eta| \leq \Delta\}$$

for a point $p \in G_l$ and number $\Delta > 0$.

Let

$$H_l(\Delta) = \{p : N(\Delta, p) \subset G_l\}.$$

Condition 1. There exists a constant $\lambda \in (0, 1)$ with the following property. For any $l \in \Lambda$ there exist $s_l \times s_l$ and $u_l \times u_l$ matrices A_l and B_l such that

$$\|A_l\| \leq \lambda \quad \text{and} \quad \|(B_l)^{-1}\| \leq \lambda \quad (3)$$

and if $p \in H_l(\Delta)$ for some $\Delta > 0$ (so that $p + (\xi, \eta) \in N(\Delta, p)$), then

$$f(p + (\xi, \eta)) = f(p) + (A_l \xi, B_l \eta). \quad (4)$$

Remark 1. We impose these simple conditions on the mapping f for the following two reasons:

- they allow us to make the proofs and estimates maximally “transparent” (of course, similar results are valid under more general hyperbolicity conditions on f in the sets G_l);
- precisely these conditions are satisfied in our main application, Theorem 2 below.

First we note that the following statement is proved by a standard reasoning (for example, it is enough to consider images under the mapping f of the “rectangles” $N(L_1 d, x_j)$, $0 \leq j < m$).

Lemma 1. *Let*

$$L_1 = \frac{1}{1 - \lambda}. \quad (5)$$

If $\{x_k : 0 \leq k \leq m\}$, where $m > 0$, is a finite d -pseudotrajectory of f (this means that inequalities (1) are satisfied for $0 \leq k \leq m - 1$) for which there exists an index $l \in \Lambda$ such that

$$x_j \subset H_l(L_1 d), \quad 0 \leq j < m,$$

then there exists a point y such that

$$f^j(y) \in N(L_1 d, x_j), \quad 0 \leq j \leq m. \quad (6)$$

Now we define geometric objects which are important in what follows.

Let $p \in G_l$, $l \in \Lambda$; introduce coordinates (ξ, η) such that p is the origin and the coordinate subspaces are parallel to S_l and U_l , respectively. Fix $\Delta_1, \Delta_2 > 0$. Consider a continuous function $\Xi(\eta)$ that maps the disk

$$\{\eta : \eta \in U_l, |\eta| \leq \Delta_1\}$$

to S_l and such that

$$|\Xi(\eta)| \leq \Delta_2, \quad |\eta| \leq \Delta_1.$$

Let D be the graph of $\Xi(\eta)$. Denote by $\mathcal{D}(\Delta_1, \Delta_2, p)$ the set of such disks D .

The following lemma is geometrically obvious.

Lemma 2. *If $p \in H_l(\Delta)$, $f(p) \in G_l$, and $D \in \mathcal{D}(\Delta_1, \Delta_2, p)$, where $\Delta_1, \Delta_2 \leq \Delta$, then $f(D)$ contains a disk D^* such that $D^* \in \mathcal{D}(\Delta_1/\lambda, \lambda\Delta_2, f(p))$.*

Remark 2. It is easily seen that in the proof of the main result we use the statements of Lemmas 1 and 2 (Lipschitz shadowing in G_l with constant L_1 and properties of the images of disks under f) plus the “transversality condition when we pass from one domain to another” (Condition 2 below). The assumed linearity of f in the domains G_l just allows us to make Lemmas 1 and 2 obvious.

Condition 2. There exist numbers $K \geq L_0 + 1$ and $\delta_0 > 0$ with the following properties. If $L_2 = L_1 + L_0 + 1$, $d \leq \delta_0$, and there exist three points p, q, r such that

$$(2.1) \quad p \in G_l \text{ and } f^2(p) \in G_m \text{ for some } l, m \in \Lambda \text{ with } l \neq m;$$

$$(2.2) \quad q \in H_l(Kd) \text{ and } r \in H_m(Kd);$$

$$(2.3) \quad |p - q| \leq L_1 d \text{ and } |f^2(p) - r| \leq L_2 d;$$

and

$$(2.4) \quad D \in \mathcal{D}(Kd, d, q),$$

then the image $f^2(D)$ contains a disk D^* such that $D^* \in \mathcal{D}(d, Kd, r)$.

Remark 3. The above condition is applied in the situation where points p and $f^2(p)$ belong to different sets G_l and G_m and we know nothing about the position of the point $f(p)$; in a sense, this condition means that the image $f^2(D)$ is “uniformly transverse” to the “stable subspace” for f at a point r that is close enough to $f^2(p)$.

Of course, an analog of this condition can be formulated for any pair of points p and $f^m(p)$, but for our main application (see Sec. 3), the present form of Condition 2 is enough.

We prove the following “conditional” theorem on Lipschitz shadowing for a mapping f satisfying the above-formulated conditions. In Theorem 1, we deal with finite d -pseudotrajectories $\{x_k : 0 \leq k \leq T\}$ of f and show that there exist δ_0 and \mathcal{L} such that any such finite d -pseudotrajectory with $d \leq \delta_0$ is $\mathcal{L}d$ -shadowed by a fragment of an exact trajectory of f . It is shown that δ_0 and \mathcal{L} depend only on f and not on the length of the pseudotrajectory. It is easily seen that if the phase space of a dynamical system generated by a

homeomorphism is locally compact, then such a “finite Lipschitz shadowing property” implies the Lipschitz shadowing property (cf. [1, Lemma 1.1.1] and the proof of Lemma 4 below).

Theorem 1. *Let $X = \{x_k : 0 \leq k \leq T\}$ be a finite d -pseudotrajectory of f with $d \leq \delta_0$ (where δ_0 is from Condition 2). Assume that there exist (not necessarily different) indices $l_0, l_1, \dots, l_t \in \Lambda$ with $l_{i+1} \neq l_i$ and integers*

$$0 = m_0 < n_0 < m_1 < n_1 < m_2 < n_2 < \dots < m_t < n_t = T,$$

where $m_{j+1} = n_j + 2$, $j = 0, \dots, t-1$, with the following properties:

(a)

$$x_k \in H_{l_j}(K_1 d), \quad m_j \leq k \leq n_j, \quad j = 0, \dots, t, \quad (7)$$

where $K_1 = K + L_1$;

(b) there exists a positive number μ for which the inequalities

$$\mu_j := n_j - m_i \geq \mu, \quad j = 0, \dots, t, \quad (8)$$

and

$$\lambda^\mu K < 1 \quad (9)$$

are satisfied.

Let

$$\mathcal{L} = L_0(L_1 + 2K) + 1. \quad (10)$$

Then there exists a point z such that

$$|f^k(z) - x_k| \leq \mathcal{L}d, \quad k = 0, \dots, T. \quad (11)$$

Remark 4. Let us emphasize that only adjacent indices l_{i+1} and l_i are assumed to be different; thus, we do not exclude the situation where the pseudotrajectory “returns” to some sets G_l several times.

In the proof of Theorem 1, we apply the following statement which is a direct corollary of Lemma 2.

Lemma 3. *Assume that for some $d > 0$ and set G_l there exists a point y and a number m such that*

$$N(Kd, f^k(y)) \subset G_l, \quad 0 \leq k \leq m, \quad (12)$$

and

$$\lambda^m K < 1. \quad (13)$$

Then for any disk $D \in \mathcal{D}(d, Kd, y)$ there exists a subset $D' \subset D$ such that

$$f^k(D') \subset N(Kd, f^k(y)), \quad 0 \leq k \leq m, \quad (14)$$

and $f^m(D')$ contains a disk $D^* \in \mathcal{D}(Kd, d, f^m(y))$.

Proof of Theorem 1. Fix a $d \leq \delta_0$. Condition (a) allows us to apply Lemma 1 to any “fragment”

$$\{x_k : m_j \leq k \leq n_j\}, \quad j = 0, \dots, t,$$

of the pseudotrajectory X and to find points y_j , $j = 0, \dots, t$, such that

$$|f^k(y_j) - x_{m_j+k}| \leq L_1 d, \quad 0 \leq k \leq \mu_j. \quad (15)$$

It follows from condition (7) that analogs of inclusions (12) in Lemma 3 are satisfied for the points y_j :

$$N(Kd, f^k(y_j)) \subset G_{l_j}, \quad 0 \leq k \leq \mu_j, \quad j = 0, \dots, t. \quad (16)$$

Since $\mu_0 = n_0 - m_0 \geq \mu$ (see (8)), it follows from (9) that condition (13) of Lemma 3 is satisfied for $y = y_0$ and $m = \mu_0$.

Let (ξ, η) be coordinates with coordinate subspaces parallel to S_{l_0} and U_{l_0} , respectively, for which y_0 is the origin.

Set

$$D_{0,0} = \{(0, \eta) : |\eta| \leq d\}.$$

Clearly, $D_{0,0} \in \mathcal{D}(d, Kd, y_0)$.

Applying Lemma 3, we find a subset D_0 of $D_{0,0}$ such that analogs of inclusions (14) are valid, i.e.,

$$f^k(D_0) \subset N(Kd, f^k(y_0)), \quad 0 \leq k \leq \mu_0,$$

and $f^{\mu_0}(D_0)$ contains a disk $D_0^* \in \mathcal{D}(Kd, d, f^{\mu_0}(y_0))$.

Let us denote $p = x_{n_0}$, $q = f^{\mu_0}(y_0)$, and $r = y_1$. It follows from (15) (with $j = 0$ and $k = n_0$) that

$$|p - q| = |x_{n_0} - f^{\mu_0}(y_0)| = |x_{n_0} - f^{n_0}(y_0)| \leq L_1 d. \quad (17)$$

Since X is a d -pseudotrajectory,

$$\begin{aligned} |f^2(p) - x_{m_1}| &= |f^2(x_{n_0}) - x_{n_0+2}| \leq |f^2(x_{n_0}) - f(x_{n_0+1})| + \\ &+ |f(x_{n_0+1}) - x_{n_0+2}| \leq (L_0 + 1)d \end{aligned}$$

(recall that L_0 is the Lipschitz constant of f). Now we estimate

$$|f^2(p) - r| \leq |f^2(p) - x_{m_1}| + |x_{m_1} - y_1| \leq (L_0 + L_1 + 1)d = L_2d \quad (18)$$

(we again refer to (15) to estimate the term $|x_{m_1} - y_1|$).

Condition 2 and estimates (17) and (18) imply that $f^2(D_0^*)$ contains a disk $D_{1,0} \in \mathcal{D}(d, Kd, y_1)$. After that, we find a subset $D_1 \subset D_{1,0}$ that has properties similar to those of D_0 , and so on.

As a result, we construct sets D_j , $j = 0, \dots, t$, such that

$$D_{j+1} \subset f^{\mu_j+2}(D_j), \quad j = 0, \dots, t-1,$$

and

$$f^k(D_j) \subset N(Kd, f^k(y_j)), \quad 0 \leq k \leq \mu_j, \quad j = 0, \dots, t. \quad (19)$$

Hence, there exists a point $z \in D_0$ such that

$$f^{m_j}(z) \in D_j, \quad j = 0, \dots, t.$$

It follows from inclusions (19) and estimates (15) that

$$|f^k(z) - x_k| \leq (L_1 + 2K)d < \mathcal{L}d, \quad m_j \leq k \leq n_j, \quad j = 0, \dots, t. \quad (20)$$

It remains to estimate the values $|f^k(z) - x_k|$ for $k = n_j + 1$. Let $z' = f^{n_j}(z)$. Then it follows from (15) that

$$\begin{aligned} |f(z') - x_{n_j+1}| &\leq |f(z') - f(x_{n_j})| + |f(x_{n_j}) - x_{n_j+1}| \leq \\ &\leq L_0(L_1 + 2K)d + d = \mathcal{L}d. \end{aligned}$$

This completes the proof of Theorem 1. \square

Remark 5. In parallel to the shadowing property, the so-called inverse shadowing property is also studied (see, for, example, [8]). It seems interesting to obtain an analog of Theorem 1 for the Lipschitz inverse shadowing. Note that the reasoning applied above in the proof of Theorem 1 cannot be directly transferred to the case of inverse shadowing.

3. Example. Consider the segment

$$I_0 = [-7/6, 4/3]$$

and a mapping $f_0 : I_0 \rightarrow I_0$ defined as follows:

$$f_0(x) = 1 + (x - 1)/2, \quad x \in [1/3, 4/3],$$

$$f_0(x) = 2x, \quad x \in (-1/3, 1/3).$$

$$f_0(x) = -1 + (x + 1)/2, \quad x \in [-7/6, -1/3].$$

Clearly, the restriction f^* of f_0 to $[-1, 1]$ is a homeomorphism of $[-1, 1]$ having three fixed points: the points $x = \pm 1$ are attracting and the point $x = 0$ is repelling (and this homeomorphism f^* is an example of the so-called “North Pole – South Pole” dynamical system; every trajectory starting at a point $x \neq 0, \pm 1$ tends to an attractive fixed point as time tends to $+\infty$ and to the repelling fixed point as time tends to $-\infty$).

Now we define a homeomorphism $f : [-1, 1] \rightarrow [-1, 1]$. For an integer $n \geq 0$, denote $\mathcal{N}_n = 2^{-(n+2)}$, and set

$$f(x) = \mathcal{N}_n f_0(\mathcal{N}_n^{-1}(x - 3\mathcal{N}_n)) + 3\mathcal{N}_n, \quad x \in (2\mathcal{N}_n, 4\mathcal{N}_n]. \quad (21)$$

This defines f on $(0, 1]$. Set $f(0) = 0$ and $f(x) = -f(-x)$ for $x \in [-1, 0)$.

Clearly, f is a homeomorphism with a nonisolated fixed point $x = 0$ (for example, every point $x = \pm 2^{-n}$ is fixed). Let us note that in a neighborhood of any fixed point (with the exception of $x = 0$), f is either linearly expanding with coefficient 2 or linearly contracting with coefficient $1/2$.

Theorem 2. *The homeomorphism f has the Lipschitz shadowing property.*

Before proving Theorem 2, we prove two auxiliary lemmas (and refer to Theorem 1 in the first of them).

In what follows, we denote by $N(d, A)$ the closed d -neighborhood of a set A .

Lemma 4. *The mapping f_0 has the Lipschitz shadowing property on I_0 .*

Proof. Let $\xi = \{x_k \in I_0 : k \in \mathbb{Z}\}$ be a d -pseudotrajectory of f_0 . In the following (very rough) estimates, we, as usual, decrease values of d , if necessary; every time, the chosen value of d is not more than the previous values. First we assume that $d \leq d_1 < 1/24$.

Note that

$$f_0(-7/6) = -13/12, \quad f_0(4/3) = 7/6.$$

Set

$$I'_0 = [-27/24, 29/24].$$

Since ξ must belong to $N(d, f_0(I_0))$, we conclude that

$$\xi \subset I'_0. \quad (22)$$

Now let us describe the possible position of ξ in I'_0 .

We note that

$$f_0(5/12) = 17/24.$$

If there exists an index k such that $|x_k| \geq 5/12$, then

$$|x_{k+i}| > 16/24 > 5/12, \quad i \geq 1.$$

Note that both f_0 and f_0^{-1} have Lipschitz constant 2. Thus, if ξ is a d -pseudotrajectory of f_0 , then ξ is a $2d$ -pseudotrajectory of f_0^{-1} .

If there exists an index k_0 such that $1/4 \leq |x_{k_0}| \leq 5/12$, then

$$|f_0^{-1}(x_{k_0})| \in [1/8, 5/24],$$

and there exists a $d_2 > 0$ such that if $d \leq d_2$, then

$$|x_k| \leq 5/24 + 2d, \quad k < k_0. \quad (23)$$

Thus, only one of the following possibilities can be realized for $d < d_2$:

- (1) $|x_k| \leq 1/4$ for $k \in \mathbb{Z}$;
- (2) $5/12 \leq |x_k| \leq 29/24$ for $k \in \mathbb{Z}$;
- (3) there exists an index k_0 such that $1/4 \leq |x_{k_0}| \leq 5/12$ and inequalities (23) hold.

In cases (1) and (2), ξ belongs to a domain in which f_0 is hyperbolic (and ξ is uniformly separated from the boundaries of the domain); by Lemma 1, there exists a $d_3 > 0$ such that if $d < d_3$, then ξ is $2d$ -shadowed by an exact trajectory of f_0 .

Consider the remaining case (3) (and let, for definiteness, $k_0 = 1$ and $x_1 > 0$; the case $x_1 < 0$ is treated similarly).

Denote $p = x_0$. Set $G_0 = [-1/3, 1/3]$ and $G_1 = [1/3, 29/24]$. Thus, $p \in G_0$.

As was mentioned, we can take $L_0 = 2$ and the statement of Lemma 1 holds for G_0 and G_1 with $L_1 = 2$.

Since $p \in [1/8 - 2d, 5/24 + 2d]$, there exists a $d_4 > 0$ such that if $d < d_4$, then

$$5/24 + 4d < 1/3 \text{ and } N(5d, f^2(p)) \subset G_1. \quad (24)$$

Take a point q such that

$$|p - q| \leq L_1 d = 2d.$$

In this case, it follows from (24) that $q \in G_0$, and, defining disks from $\mathcal{D}(\Delta_1, \Delta_2, q)$, we must take $S_0 = \{0\}$ and $U_0 = \mathbb{R}$. Thus, if $K > 0$, then the set $\mathcal{D}(Kd, d, q)$ contains precisely one disk

$$D = [q - Kd, q + Kd].$$

If $K > 2$, then D contains the disk

$$D' = [p - K'd, p + K'd],$$

where $K' = K - 2$.

Clearly, $f^2(D')$ contains the disk

$$D'' = [f^2(p) - K'd/4, f^2(p) + K'd/4].$$

If a point r satisfies the inequality

$$|f^2(p) - r| \leq L_2 d = 5d,$$

it follows from the second inclusion in (24) that $r \in G_1$, and, defining disks from $\mathcal{D}(\Delta_1, \Delta_2, r)$, we must take $U_1 = \{0\}$ and $S_1 = \mathbb{R}$. Thus, the set $\mathcal{D}(d, Kd, r)$ consists of points r' such that $|r' - r| \leq d$.

It follows that Condition 2 is satisfied if $d_0 \leq d_4$ and $K'/4 \geq 6$. Thus, it is enough to take $K = 26$.

Now, when L_0 , L_1 , and K are fixed, it is easily seen that there exists a $d_0 > 0$ such that if $d \leq d_0$, then

$$x_k \in H_0(K_1 d) = [-1/3 + K_1 d, 1/3 - K_1 d], \quad k \leq 0, \quad (25)$$

and

$$x_k \in H_1(K_1 d) = [1/3 - K_1 d, 29/24 + K_1 d], \quad k \geq 2. \quad (26)$$

To apply Theorem 1, we fix a natural number n and change indices of points of the d -pseudotrajectory $\xi = \{x_k\}$ to obtain a d -pseudotrajectory $\xi^{(n)} = \{x_k^{(n)}\}$, where

$$x_k^{(n)} = x_{k-n}, \quad k \in \mathbb{Z}.$$

Setting $m_0 = 0$, $n_0 = n$, $m_1 = n + 2$, $m_2 = 2n + 2$, we conclude from inclusions (25) and (26) that

$$x_k^{(n)} \in H_0(K_1 d), \quad m_0 \leq k \leq n_0,$$

and

$$x_k^{(n)} \in H_1(K_1 d), \quad m_1 \leq k \leq n_1.$$

Thus, condition (a) of Theorem 1 is satisfied.

It is clear that if $2^{-n-1}K < 1$, then condition (b) of Theorem 1 is satisfied as well.

By Theorem 1, there exists a point $z^{(n)}$ such that

$$|f^k(z^{(n)}) - x_k^{(n)}| \leq \mathcal{L}d, \quad 0 \leq k \leq 2n + 2.$$

Hence, if $\zeta^{(n)} = f^n(z^{(n)})$, then

$$|f^k(\zeta^{(n)}) - x_k| \leq \mathcal{L}d, \quad -n \leq k \leq n+2. \quad (27)$$

Let ζ be a limit point of the sequence $\zeta^{(n)}$. Passing to the limit as $n \rightarrow \infty$ in (26) and taking into account that f is a homeomorphism (so that any f^k is continuous), we see that

$$|f^k(\zeta) - x_k| \leq \mathcal{L}d, \quad k \in \mathbb{Z}.$$

□

The following statement is almost obvious.

Lemma 5. *Let g be a mapping of a segment J and let numbers $M > 0$ and m be given. Consider the mapping*

$$g'(y) = M^{-1}g(M(y - m)) + m$$

on the set

$$J' = \{y : M(y - m) \in J\}.$$

If g has the Lipschitz shadowing property with constants \mathcal{L}, d_0 , then g' has the Lipschitz shadowing property with constants $\mathcal{L}, M^{-1}d_0$.

Proof. First we note that if $\{y_k\}$ is a d -pseudotrajectory of g' with $d \leq d_0/M$ and $x_k = M(y_k - m)$, then

$$g(x_k) - x_{k+1} = M(g'(y_k) - y_{k+1}).$$

Hence, $\{x_k\}$ is an Md -pseudotrajectory of g .

Since $Md \leq d_0$, there exists a point p such that

$$|g^k(p) - x_k| \leq \mathcal{L}Md.$$

Set $p' = M^{-1}p + m$. Then, obviously,

$$|(g')^k(p') - y_k| = M^{-1}|g^k(p) - x_k| \leq \mathcal{L}d.$$

□

Let us prove Theorem 2.

For a natural n , define the segment

$$I_n = [\alpha_n, \beta_n] = [11\mathcal{N}_n/6, 13\mathcal{N}_n/3]$$

and note that formula (21) defining f for $x \in (2\mathcal{N}_n, 4\mathcal{N}_n]$ is, in fact, valid for $x \in I_n$.

It follows from the equalities

$$f(\alpha_n) = 23\mathcal{N}_n/12, \quad f(\beta_n) = 25\mathcal{N}_n/6$$

that $f(I_n) \subset N(\mathcal{N}_n/12, I_n)$. Thus, if $d < \delta(n) = \mathcal{N}_n/12$ and $\{x_k\}$ is a d -pseudotrajectory of f that intersects I_n , then $\{x_k\} \subset I_n$.

Let d_0 and \mathcal{L} be the constants of Lipschitz shadowing for f_0 given by Lemma 4. Since $d_0 < 1/12$, it follows from Lemma 5 that if $\{x_k\}$ is a d -pseudotrajectory of f that intersects I_n for some $n > 0$, then $\{x_k\}$ is $\mathcal{L}d$ -shadowed. Of course, a similar statement holds for the segments $I'_n = [-\beta_n, -\alpha_n]$.

To complete the proof, consider a d -pseudotrajectory $\xi = \{x_k\}$ of f with $d \leq d_0$ and find the maximal n_0 for which $d < \delta(n_0)$. Note that then

$$d \geq \mathcal{N}_{n_0+1}/12.$$

If ξ intersects one of the segments I_n or I'_n with $n \leq n_0$, then everything is proved.

Otherwise,

$$|x_k| \leq \alpha(n_0) = 11\mathcal{N}_{n_0+1}/3 \leq 44d,$$

and ξ is $44d$ -shadowed by the rest point $x = 0$. \square

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References

1. S. Yu. Pilyugin, *Shadowing in dynamical systems*, Lecture Notes in Mathematics, Springer, **1706** (1999).
2. K. Palmer, *Shadowing in dynamical systems. Theory and applications*, Kluwer (2000).
3. S. Yu. Pilyugin, Theory of pseudo-orbit shadowing in dynamical systems, *Differential Equations*, **47** (2011), 1929-1938.

4. S. Yu. Pilyugin and S. B. Tikhomirov, Lipschitz shadowing implies structural stability, *Nonlinearity*, **23** (2010), 2509-2515.
5. S. B. Tikhomirov, Hölder shadowing on finite intervals, *Ergodic Theory Dynam. Systems* (accepted). arXiv:1106.4053v2.
6. A. A. Petrov and S. Yu. Pilyugin, Shadowing near nonhyperbolic fixed points, *Discrete Contin. Dynam. Syst.*, **34** (2014), 3761-3772.
7. R. Mañé, *Characterizations of AS diffeomorphisms*, Lecture Notes in Mathematics, Springer, **597** (1977), 389-394.
8. S. Yu. Pilyugin, Inverse shadowing by continuous methods, *Discrete Contin. Dynam. Syst.*, **8** (2002), 29-38.